AIAA JOURNAL
Vol. 35, No. 4, April 1997

Analysis of Uncertain Structural Systems Using Interval Analysis

S. S. Rao*
Purdue University, West Lafayette, Indiana 47907-1288
and
L. Berke†
NASA Lewis Research Center, Cleveland, Ohio 44135

The imprecision or uncertainty present in many engineering analysis/design problems can be modeled using probabilistic, fuzzy, or interval methods. This work considers the modeling of uncertain structural systems using interval analysis. By representing each uncertain input parameter as an interval number, a static structural analysis problem can be expressed in the form of a system of linear interval equations. In addition to the direct and Gaussian elimination-based solution approaches, a combinatorial approach (based on an exhaustive combination of the extreme values of the interval numbers) and an inequality-based method are presented for finding the solution of interval equations. The range or interval of the solution vector (response parameters) is found to increase with increasing size of the problem in all of the methods. An interval–truncation approach is proposed to limit the growth of intervals of response parameters so that realistic and accurate solutions can be obtained in the presence of large amounts of uncertainty. Numerical examples are presented to illustrate the computational aspects of the methods and also to indicate the importance of the truncation approach in practical problems. The utility of interval methods in predicting the extreme values of the response parameters of structures is discussed.

I. Introduction

ALL engineering design problems involve imprecision or approximation or uncertainty to varying degrees. As an example, consider a beam whose length may be stated in different ways as follows: "lies between 3.4 and 3.6 m," or "about 3.5 m," or "has a mean value of 3.5 m and a standard deviation of 0.05 m, and follows normal distribution." Depending on the nature of imprecision, the analysis/design of the system can be conducted using interval analysis,[1,2] fuzzy theory,[3] or a probabilistic approach.[4] In interval analysis, the uncertain parameter is denoted by a simple range. In addition to the range, if a preference function is used to describe the desirability of using different values within the range, fuzzy theory can be used. In fuzzy theory, the imprecision is interpreted as the designer's choice to use a particular value for the uncertain parameter. On the other hand, if the uncertain parameter is described as a random variable following a specified probability distribution, the probabilistic approach can be used. It can be seen that when information about an uncertain parameter in the form of a preference or probability function is not available, the interval analysis can be used most conveniently.

The following examples indicate typical situations in which an uncertain parameter can be modeled as an interval number.

1) In design and manufacture, if a geometric parameter x is subjected to tolerances as $x \pm \Delta x$, then the parameter has to be treated as an interval number as $\bar{x} = [x - \Delta x, x + \Delta x]$.

2) In engineering analysis, certain parameters such as the magnitude of wind load (P) may be known to vary over a range P_1 to P_2; but the probability distribution of P may not be known. In such a case, the load can be treated as an interval number as $P = [P_1, P_2]$.

3) In the sensitivity analysis of a system, we might be interested in finding the influence of changing a parameter over a specified range, $x \pm \Delta x$. In such a case, the influence of changes in the independent variable x on a dependent variable (response quantity) f can be represented by an interval as $f = [f - \Delta f, f + \Delta f]$ where $\pm\Delta f$ denotes the variations in f caused by the changes $\pm\Delta x$ in x.

4) Many engineering analyses use a Taylor-series approximation. The absolute value of the error involved in using only a finite number of terms in the Taylor series is always known so that the actual value of the function being represented becomes an interval. For example, if $f(x)$ is expanded around $x*$ using the series

$$\tilde{f}(x) = \tilde{f}(x* + \Delta x) \approx f(x*) + f'(x*)\Delta x + f''(x*)\frac{(\Delta x)^2}{2!} + \dots + f^{(k)}(x*)\frac{(\Delta x)^k}{k!} \tag{1}$$

the error is given by

$$e = f^{(k+1)}(\xi)\frac{(\Delta x)^{k+1}}{(k+1)!} \qquad x* < \xi < x* + \Delta x \tag{2}$$

so that the actual value of $f(x)$ lies in the interval $f(x) = [\tilde{f}(x) - e, \tilde{f} + e]$.

5) The performance characteristics of most engineering systems vary during their lifetimes because of aging, creep, wear, and changes in operating conditions. The design of such systems should take care of the intervals of deviation of variables from their nominal values.

Although the analysis/design problems are to be solved using intervals for the design parameters, most traditional methods consider only nominal values for simplicity and convenience. A fuzzy-analysis-based approach was presented by Rao and Sawyer[5] for the description of systems containing information and features that are vague, imprecise, qualitative, linguistic, or incomplete. In particular, a fuzzy finite element method was developed for predicting the response of systems governed by linear systems of equations. In the reliability analysis of structures using stress–strength interference theory, an estimation of the failure probability of the structure is based on a precise knowledge of the exact probability distributions of the random parameters.[6] Even then, the numerical computation of the extreme values of the induced stress requires the modeling of tail portions of the probability distributions of the random variables, which is very difficult in practice. It can be seen from the literature that interval analysis methods were not used in engineering applications.

This paper presents methods that can be used for the analysis of engineering systems for which the input parameters are given as

Received Sept. 6, 1995; accepted for publication June 18, 1996.

*Professor, School of Mechanical Engineering. Member AIAA.

†Structures Division. Associate Fellow AIAA.

interval numbers. Because most engineering systems can be analyzed using the finite element method, the problem considered in this work can be stated as follows: Given a system of linear equations that describe the response of a discretized structural model in the form

$$[A(\boldsymbol{p})]\boldsymbol{X} = \boldsymbol{B}(\boldsymbol{p}) \tag{3}$$

and

$$\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{X}) \tag{4}$$

where $[A]$ is the stiffness matrix of the system, $\boldsymbol{p}$ is the vector of uncertain input parameters described as interval numbers as

$$\boldsymbol{p} = \begin{Bmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{Bmatrix} = \begin{Bmatrix} (\underline{p}_1, \bar{p}_1) \\ (\underline{p}_2, \bar{p}_2) \\ \vdots \\ (\underline{p}_m, \bar{p}_m) \end{Bmatrix} \tag{5}$$

where a bar below (above) a symbol denotes a lower (upper) bound, $\boldsymbol{X}$ is the nodal displacement vector of the discrete structural model, $\boldsymbol{B}$ is the vector of nodal forces (which might depend on the uncertain parameters), and $\boldsymbol{R}$ is the vector of response quantities, such as element stresses and strains, which can be expressed in terms of the nodal displacements, $\boldsymbol{X}$, of the structure. Because $\boldsymbol{R}$ implicitly depends on $\boldsymbol{p}$, there will be a range for each component of $\boldsymbol{R}$ so that

$$\boldsymbol{R} = [\underline{\boldsymbol{R}}, \bar{\boldsymbol{R}}] = \{(\underline{r}_i, \bar{r}_i)\} \tag{6}$$

The problem is to find the range of each response quantity $(\underline{r}_i, \bar{r}_i)$ due to the uncertainty present in the input parameters.

The solution of linear interval equations and the subsequent computation of the response parameters, Eqs. (3) and (4), based on direct and Gaussian elimination-based methods is discussed in this work. Alternative techniques that find the bounds on the solution using a combinatorial approach and certain inequalities are proposed. It has been observed that the widths (intervals) of the response parameters predicted become wider than the true widths with an increase in the size of the problem. To avoid the unnecessary growth of the intervals of the response parameters, a computationally efficient approximation procedure, termed the truncation method, is presented. The computational aspects and the results given by the various methods are discussed with the help of numerical examples.

II. Statement of the Problem

Most linear structural analysis problems are solved using the finite element method wherein the governing equations are represented as a system of algebraic equations:

$$[A]\boldsymbol{X} = \boldsymbol{B} \tag{7}$$

where $[A] = [a_{ij}]$, $\boldsymbol{X} = \{x_i\}$, and $\boldsymbol{B} = \{b_i\}$. If the system parameters are imprecise, the coefficients a_{ij} and the right-hand-side constants b_i in Eq. (7) are not sharply defined. In such a case, each of the coefficients and constants is represented by a closed interval as

$$a_{ij} = [\underline{a}_{ij}, \bar{a}_{ij}]; \qquad i, j = 1, 2, \ldots, n \tag{8}$$

$$b_i = [\underline{b}_i, \bar{b}_i]; \qquad i = 1, 2, \ldots, n \tag{9}$$

Equation (7) also can be written as

$$([\underline{A}], [\bar{A}])\boldsymbol{X} = (\underline{\boldsymbol{B}}, \bar{\boldsymbol{B}}) \tag{10}$$

The solution of Eq. (7) is understood to mean

$$\boldsymbol{X} = \left\{\boldsymbol{X}^0 \,\middle|\, [A^0]\boldsymbol{X}^0 = \boldsymbol{B}^0, [A^0] \in [A] = ([\underline{A}], [\bar{A}]), \boldsymbol{B}^0 \in \boldsymbol{B} = (\underline{\boldsymbol{B}}, \bar{\boldsymbol{B}})\right\} \tag{11}$$

where all $[A^0] \in [A]$ are assumed to be nonsingular. Equation (11) denotes a set of solutions (solution domain) in $\boldsymbol{X}$ space such that for every $[A^0] \in [A]$ and every $\boldsymbol{B}^0 \in \boldsymbol{B}$, there exists an $\boldsymbol{X}^0$ that satisfies the relation $[A^0]\boldsymbol{X}^0 = \boldsymbol{B}^0$. Note that the solution set, given by Eq. (11), may be empty in some cases; that is, given some $[A]$ and $\boldsymbol{B}$, there may not be an $\boldsymbol{X}$ that satisfies Eq. (7).

The solution set and its hull of Eq. (7) have been investigated by several researchers.[7,8] For systems governed by Eq. (7), two types of problems can be identified.[9] In the first type of problems, all possible vectors $\boldsymbol{B}$ are sought for specified interval matrix $[A]$ and interval vector $\boldsymbol{X}$. This gives the range of $\boldsymbol{B}$. In the second type of problems, we seek to find the interval vector $\boldsymbol{X}$ for specified interval matrix $[A]$ and interval vector $\boldsymbol{B}$. This gives the domain of solution $\boldsymbol{X}$.

The objective of this work is to investigate the solution methods for the second type of problem. The direct approach, the Gaussian elimination technique, a combinatorial approach, a bounding technique (based on certain inequalities), and a truncation approach are presented for the solution of Eq. (7). Although the direct method, the Gaussian elimination approach, and the inequalities are well established for crisp problems, their application to interval problems, particularly to those arising in engineering analyses, is new. Furthermore, the truncation technique presented in this work represents a completely new contribution. The numerical results given by the various methods are compared for typical structural analysis problems. Although the numerical examples are very simple, the interval analysis methods presented are expected to be applicable to other types of engineering problems whose behavior is governed by systems of linear equations.

III. Interval Analysis Using Direct and Gaussian Elimination Techniques

For simple systems (where the number of equations is two or three), the solution of Eq. (7) can be expressed in terms of the elements of the matrix $[A]$ and the vector $\boldsymbol{B}$. In such cases, the response of the system (such as element stresses) also can be determined using the interval arithmetic operations given in Appendix A. However, the widths or intervals of the responses so computed will be larger than the actual widths.

The solution of Eq. (7) also can be found using the Gaussian elimination method. For this, let $[D]$ be an interval matrix whose elements contain ranges of the corresponding elements of $[A]^{-1}$ in Eq. (7). Then the product of $[D]$ and the interval vector $\boldsymbol{B}$ gives the vector $\boldsymbol{X}$ whose components represent an interval that contains the possible values of the corresponding components of the solutions of Eq. (7) when the elements of $\boldsymbol{B}$ are allowed to vary over the respective specified ranges. When exact interval arithmetic is used in the computations, we obtain an interval matrix, $[A]^{-1}$, whose elements are wider than the corresponding ranges of the exact inverse matrix. The computations are shown for a sample problem in Appendix B.

There is another practical difficulty associated with the implementation of the Gaussian elimination method for the solution of interval equations. This is related to reducing the coefficients of off-diagonal variables to zero during pivot operations. The basic arithmetic operations, related to interval numbers and interval matrices, are summarized in Appendix A. Because the subtraction of an interval number from itself does not result in zero (for example, if $A = [\underline{a}, \bar{a}]$, then $A - A = [\underline{a}, \bar{a}] - [\underline{a}, \bar{a}] = [\underline{a} - \bar{a}, \bar{a} - \underline{a}] \neq [0, 0]$), the Gaussian elimination cannot be implemented accurately.

IV. Analytical Developments

A. Combinatorial Approach

If $f(x_1, x_2, \ldots, x_n)$ denotes an arbitrary (monotonic) function of the imprecise parameters $x_1, x_2, \ldots, x_n$, the range of f can be determined by considering all possible combinations of the endpoints of the imprecise parameters. For this, let the parameter x_i be denoted as an interval number as

$$x_i = \left[x_i^{(1)}, x_i^{(2)}\right] = [\underline{x}_i, \bar{x}_i]; \qquad i = 1, 2, \ldots, n \tag{12}$$

Then all possible values of f can be determined as

$$f_r = f\left(x_1^{(i)}, x_2^{(j)}, \ldots, x_n^{(k)}\right)$$

$$i = 1, 2; \quad j = 1, 2; \quad \ldots; \quad k = 1, 2; \quad r = 1, 2, \ldots, 2^n \tag{13}$$

where f_r denotes the value of f for a particular combination of the endpoints of the intervals of $x_1, x_2, \ldots, x_n$. The function f can be represented as an interval number as

$$f = [\underline{f}, \bar{f}] = \left[\min_{r=1,2,\ldots,2^n}(f_r), \max_{r=1,2,\ldots,2^n}(f_r)\right] \tag{14}$$

Equations (3) and (4) are based on linear structural analysis and hence Eq. (14) always gives the correct extreme values of the response parameters such as nodal displacements and element stresses.

B. Bounds on Solution Using Inequalities

Any system of interval equations of the form

$$[A]\boldsymbol{X} = \boldsymbol{B} \tag{15}$$

with $[A] = ([\underline{A}], [\bar{A}])$ and $\boldsymbol{B} = (\underline{\boldsymbol{B}}, \bar{\boldsymbol{B}})$ can be rewritten as

$$([A^0] - [\Delta A], [A^0] + [\Delta A])\boldsymbol{X} = (\boldsymbol{B}^0 - \Delta\boldsymbol{B}, \boldsymbol{B}^0 + \Delta\boldsymbol{B}) \tag{16}$$

where

$$[A^0] = \tfrac{1}{2}([\underline{A}] + [\bar{A}]), \qquad [\Delta A] = \tfrac{1}{2}([\bar{A}] - [\underline{A}]) \geq [0] \tag{17}$$

$$\boldsymbol{B}^0 = \tfrac{1}{2}(\underline{\boldsymbol{B}} + \bar{\boldsymbol{B}}), \qquad \Delta\boldsymbol{B} = \tfrac{1}{2}(\bar{\boldsymbol{B}} - \underline{\boldsymbol{B}}) \geq 0 \tag{18}$$

It can be observed that 1) $-[\Delta A]|\boldsymbol{X}|$ is always less than or equal to both $-[\Delta A]\boldsymbol{X}$ and $[\Delta A]\boldsymbol{X}$ and 2) $[\Delta A]|\boldsymbol{X}|$ is always greater than or equal to both $-[\Delta A]\boldsymbol{X}$ and $[\Delta A]\boldsymbol{X}$, where

$$|\boldsymbol{X}| = \begin{Bmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{Bmatrix}$$

The satisfaction of the following inequality is necessary and sufficient for $\boldsymbol{X}$ to be a solution of Eq. (15) (Refs. 10 and 11):

$$[\Delta A]|\boldsymbol{X}| + \Delta\boldsymbol{B} \geq |[A]\boldsymbol{X} - \boldsymbol{B}| \tag{19}$$

The inequality (19) is equivalent to

$$[A]\boldsymbol{X} - [\Delta A]|\boldsymbol{X}| \leq \bar{\boldsymbol{B}} \tag{20}$$

and

$$[A]\boldsymbol{X} + [\Delta A]|\boldsymbol{X}| \geq \underline{\boldsymbol{B}} \tag{21}$$

All vectors $\boldsymbol{X}$ that satisfy the inequalities (20) and (21) constitute the solution domain of Eq. (15). Equations (20) and (21) denote a set of $2n$ inequalities, which can be expressed as

$$\Delta\boldsymbol{a}_i^T|\boldsymbol{X}| + \Delta B_i \geq \boldsymbol{a}_i^T\boldsymbol{X} - B_i; \qquad i = 1, 2, \ldots, n \tag{22}$$

and

$$\Delta\boldsymbol{a}_i^T|\boldsymbol{X}| + \Delta B_i \geq -(\boldsymbol{a}_i^T\boldsymbol{X} - B_i); \qquad i = 1, 2, \ldots, n \tag{23}$$

or

$$l_i^+(\boldsymbol{X}) = \boldsymbol{a}_i^T\boldsymbol{X} - B_i - \Delta\boldsymbol{a}_i^T|\boldsymbol{X}| - \Delta B_i \leq 0; \qquad i = 1, 2, \ldots, n \tag{24}$$

and

$$l_i^-(\boldsymbol{X}) = \boldsymbol{a}_i^T\boldsymbol{X} - B_i + \Delta\boldsymbol{a}_i^T|\boldsymbol{X}| + \Delta B_i \geq 0; \qquad i = 1, 2, \ldots, n \tag{25}$$

where B_i and ΔB_i denote the ith components of $\boldsymbol{B}$ and $\Delta\boldsymbol{B}$, respectively, $\boldsymbol{a}_i$ and $\Delta\boldsymbol{a}_i$ represent the ith rows of the matrices $[A]$ and $[\Delta A]$, respectively, and the superscript T indicates the transpose. It can be seen that the vector $\boldsymbol{X}$ is restricted to the convex polyhedron bounded by the hyperplanes $l_i^+ = 0$ and $l_i^- = 0$. Because there are only n unknowns in the vector $\boldsymbol{X}$, at any vertex of the polyhedron, the equality sign holds only for n of the $2n$ conditions of Eqs. (24) and (25). It can be observed that the hyperplanes $l_i^+ = 0$ and $l_i^- = 0$ lie symmetrically with respect to the hyperplane $\boldsymbol{a}_i^T\boldsymbol{X} - B_i = 0$. As such, they cannot intersect in the orthant containing $\boldsymbol{X}^0$ where $\boldsymbol{X}^0$ satisfies the equations $[A^0]\boldsymbol{X}^0 = \boldsymbol{B}^0$ (Ref. 10). This implies that

$$l_i^+ = 0 \qquad \text{and} \qquad l_i^- > 0 \tag{26}$$

or

$$l_i^+ < 0 \qquad \text{and} \qquad l_i^- = 0 \tag{27}$$

Thus the 2^n vertices of the polyhedron are characterized by the fact that the equality sign holds only for one of the equations (24) and (25).

If all of the admissible vectors $\boldsymbol{X}$ do not lie in the same orthant, the solution of Eq. (15) is given by the union of convex polyhedra, which may not be convex. To determine the vertices of the solution set of Eq. (15), we need to find the minimum/maximum values of the various components x_j of the solution. This can be achieved by solving the following linear programming problems[10]:

$$\text{minimize/maximize } x_j$$

subject to

$$\begin{aligned} l_i^+(\boldsymbol{X}) &\leq 0; \qquad i = 1, 2, \ldots, n \\ -l_i^-(\boldsymbol{X}) &\leq 0; \qquad i = 1, 2, \ldots, n \end{aligned} \tag{28}$$

It can be seen that the problem of Eq. (28) with minimization (maximization) gives the lower (upper) bound on the interval of x_j. Thus the solution of $2n$ linear programming problems provides the solution of Eq. (15).

C. Interval Analysis with Truncation

From the definitions of interval arithmetic operations, given by Eq. (A1), it can be seen that the evaluation of any rational expression in interval arithmetic will lead to the exact range of values of the real rational function defined by the expression if each variable occurs only once in the expression.[1] Because it is not possible to rewrite complicated and/or implicit expressions in which a number of interval variables appear several times, there is no simple way to obtain the exact intervals of functions involved in most engineering applications. Thus it can be expected that the width of the interval of a function grows with the number of interval variables and the number of arithmetic operations (or size of the problem) involved.

To limit the growth of intervals of response parameters for large amounts of uncertainties, a truncation procedure is proposed. This procedure can be described as follows. Let $a = [\underline{a}, \bar{a}]$ and $b = [\underline{b}, \bar{b}]$ denote the input interval variables and $c = [\underline{c}, \bar{c}]$ the result of an interval arithmetic operation. If the central values of the variables $a_0 = (\underline{a} + \bar{a})/2$ and $b_0 = (\underline{b} + \bar{b})/2$ are used, the corresponding arithmetic operation on the central numbers leads to the result c_0. Let c_0 be called the crisp (or central) result that can be obtained when the variability is absent in the input variables. If c_0 is close to zero, no truncation is used. On the other hand, if c_0 is not close to zero, then the relative deviation (Δ) of the interval, $\bar{c} - \underline{c}$, from the central value is computed as

$$\Delta = \Delta_1 + \Delta_2 \tag{29}$$

where

$$\Delta_1 = \left|\frac{\underline{c} - c_0}{c_0}\right|, \qquad \Delta_2 = \left|\frac{c_0 - \bar{c}}{c_0}\right| \tag{30}$$

Because the deviation Δ is expected to be larger than the true deviation, the maximum permissible relative width of c is specified as $2t$. The value of t can be specified easily on the basis of the known deviations of the input variables a and b. For example, the value of t can be specified to be equal to the maximum of the relative deviations of the input parameters a and b from their respective central values. The interval number $c = [\underline{c}, \bar{c}]$ is truncated to obtain the result $c \approx [\underline{d}, \bar{d}]$ as follows:

$$\underline{d} = \underline{c}, \qquad \bar{d} = \bar{c} \qquad \text{if} \qquad \Delta \leq 2t \tag{31}$$

(if both Δ_1 and Δ_2 are smaller than or equal to the permissible deviation t, no truncation is used),

$$\underline{d} = c_0 + t(\underline{c} - c_0), \qquad \bar{d} = c_0 + t(\bar{c} - c_0) \qquad \text{if} \qquad \Delta > 2t \tag{32}$$

(if both Δ_1 and Δ_2 are larger than the permissible deviation t, $[(c - c_0)/c_0]$ is truncated at $-t$ and $+t$),

$$\underline{d} = \underline{c}, \qquad \bar{d} = c_0 + t(\bar{c} - c_0) \qquad \text{if} \qquad \Delta_1 \leq t \quad \text{and} \quad \Delta_2 > t \tag{33}$$

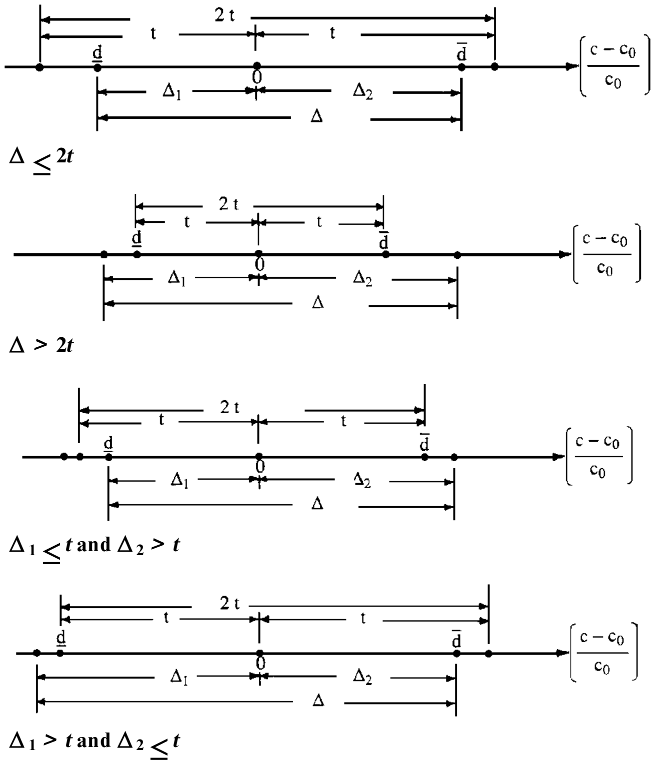

Fig. 1 Truncation procedure.

(if Δ_1 is smaller than or equal to t and Δ_2 is larger than the permissible value t, c is not truncated on the left side of c_0 but $[(c - c_0)/c_0]$ is truncated at t),

$$\bar{d} = \bar{c}, \qquad \underline{d} = c_0 + t(\underline{c} - c_0) \quad \text{if} \quad \Delta_1 > t \quad \text{and} \quad \Delta_2 \leq t \tag{34}$$

Note that if Δ_2 is smaller than or equal to t and Δ_1 is larger than the permissible value t, c is not truncated on the right-hand side of c_0 but $[(c - c_0)/c_0]$ is truncated at $-t$.

The significance of Eqs. (31–34) is shown graphically in Fig. 1. It has been observed that a value of t equal to the maximum relative deviation (about the central value) of the input interval parameters gives accurate intervals of the response (output) parameters. In addition, it was found that the exact value of t is not very critical as long as its value is equal to or somewhat larger than the largest relative deviation present in the uncertain input parameters.

V. Numerical Examples

A. Example 1: Equations with Arbitrary Intervals

To illustrate the analysis methods presented, the following linear interval equations are considered:

$$\begin{bmatrix} [3, 6] & [-3, 1.5] \\ [-1.5, 3] & [3, 6] \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} [-4, 4] \\ [-4, 4] \end{Bmatrix} \tag{35}$$

The solutions given by the various methods are described below.

1. Combinatorial Solution

As stated earlier, the combinatorial approach identifies only the extreme vertices of the solution domain. For the system of Eqs. (35), the combinatorial approach gives the corner points C_i ($i = 1, 2, 3, 4$) shown in Fig. 2. Thus the square region bounded by the points C_i($i = 1, 2, 3, 4$) defines the solution domain of the system according to the combinatorial approach.

2. Inequality-Based Solution

Because the problem is simple with the number of equations $n = 2$, the vertices of the solution domain can be determined by solving the inequalities (26) and (27) using a trial-and-error procedure. The inequalities (20) and (21) corresponding to Eqs. (35) define the solution domain shown in Fig. 2 [can be identified by plotting Eqs. (20) and (21)].

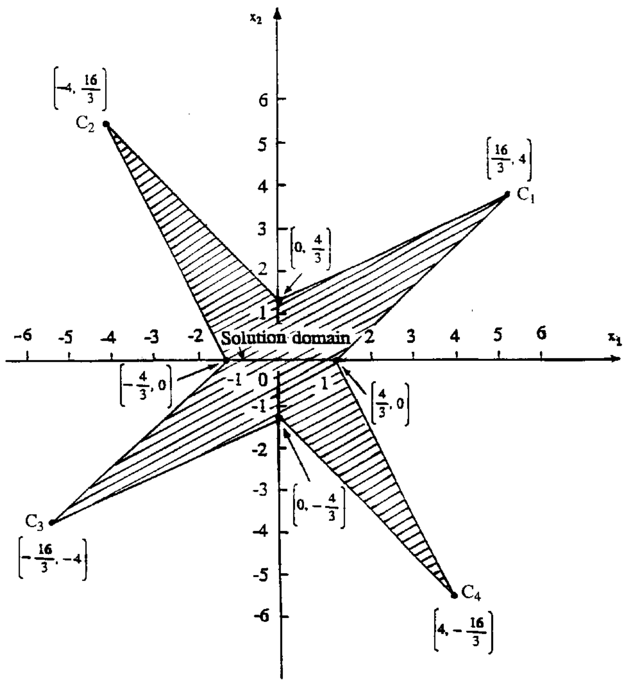

Fig. 2 Solution domain of Eqs. (35).

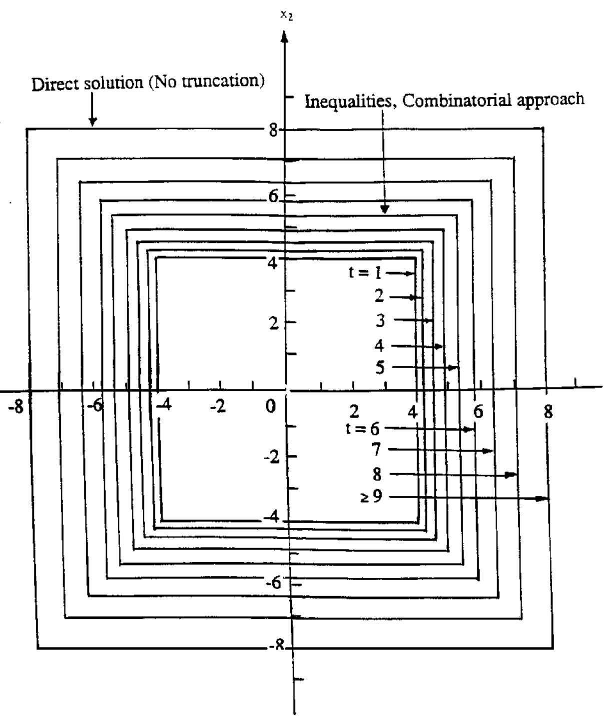

Fig. 3 Convergence of solution domain of Eqs. (35).

3. Truncation Solution

To illustrate the truncation approach, Eqs. (35) are solved using a direct method. The convergence of the solutions obtained with different values of the truncation parameter t are shown in Fig. 3. It can be seen that, as the value of t increases, the results converge to those of the normal interval analysis (with no truncation). Note that the widths of the coefficients and constants involved in Eqs. (35), relative to their respective middle values, are given by

$$a_{11}, a_{22}: \left|\frac{6 - 3}{4.5}\right| = 0.6667, \qquad a_{12}: \left|\frac{3 - (-1.5)}{0.75}\right| = 6.0$$
$$a_{21}: \left|\frac{1.5 - (-3)}{-0.75}\right| = 6.0, \qquad b_1, b_2: \left|\frac{4 - (-4)}{0}\right| = \infty \tag{36}$$

It is found that, by setting the value of t approximately equal to the maximum relative width encountered in the problem data (ignoring infinity), we can obtain an accurate solution of the problem. In the case of Eqs. (35), the maximum relative width is 6.0 (for a_{12} and a_{21}) and hence the use of $t \approx 6.0$ gives a solution close to the one predicted by the combinatorial approach. As can be seen from Fig. 3, with larger values of t (including, theoretically, infinity), the truncation procedure gives results that converge to those of the normal interval analysis (with no truncation).

B. Example 2: Stepped Bar

The stress analysis of a stepped bar under axial load, as shown in Fig. 4, is considered and the results given by the various interval analysis methods are described below.

1. *Direct Solution*

If $A_1 = [\underline{A}_1, \bar{A}_1]$ and $A_2 = [\underline{A}_2, \bar{A}_2]$ are the cross-sectional areas, $E_1 = [\underline{E}_1, \bar{E}_1]$ and $E_2 = [\underline{E}_2, \bar{E}_2]$ are the Young's moduli, $l_1 = [\underline{l}_1, \bar{l}_1]$ and $l_2 = [\underline{l}_2, \bar{l}_2]$ are the lengths of the two sections of the bar, and $p_2 = [\underline{p}_2, \bar{p}_2]$ is the axial load applied at the end, the equilibrium equations can be stated as

$$[K]\boldsymbol{X} = \boldsymbol{P} \tag{37}$$

where $[K] = [k_{ij}]$ is the 2×2 stiffness matrix, $\boldsymbol{X} = \{x_i\}$ is the two-component displacement vector, and $\boldsymbol{P} = \{p_i\}$ is the two-component load vector. The maximum displacements (x_1 and x_2) and the stresses (σ_1 and σ_2) induced in the two sections of the bar can be expressed, using a direct method, as

$$x_1 = \frac{p_2 k_{12}}{k_{12}^2 - k_{11}k_{22}}, \qquad x_2 = -\frac{k_{11}x_1}{k_{12}} \tag{38}$$

$$\sigma_1 = \frac{x_1 E_1}{l_1}, \qquad \sigma_2 = \frac{(x_2 - x_1)E_2}{l_2} \tag{39}$$

where

$$k_{11} = \left[\frac{\underline{A}_1\underline{E}_1}{\bar{l}_1} + \frac{\underline{A}_2\underline{E}_2}{\bar{l}_2}, \frac{\bar{A}_1\bar{E}_1}{\underline{l}_1} + \frac{\bar{A}_2\bar{E}_2}{\underline{l}_2}\right] \tag{40}$$

$$k_{12} = k_{21} = \left[-\frac{\bar{A}_2\bar{E}_2}{\underline{l}_2}, -\frac{\underline{A}_2\underline{E}_2}{\bar{l}_2}\right] \tag{41}$$

$$k_{22} = \left[\frac{\underline{A}_2\underline{E}_2}{\bar{l}_2}, \frac{\bar{A}_2\bar{E}_2}{\underline{l}_2}\right] \tag{42}$$

Numerical results are obtained with the following base data: $A_1^0 = 2$ in.^{2}, $A_2^0 = 1$ in.^{2}, $E_1^0 = E_2^0 = 30 \times 10^6$ psi, $l_1^0 = 10$ in., $l_2^0 = 5$ in., $p_1^0 = 0$, and $p_2^0 = 100$ lb, where the superscript zero to a symbol denotes a base value. The interval parameter x, for example, is taken as $x = [\underline{x}, \bar{x}]$ with $\underline{x} = x^0 - \Delta x$, $\bar{x} = x^0 + \Delta x$, and Δx is a certain percentage of x. The stresses induced in sections 1 and 2 when the intervals of areas of cross section, Young's moduli, lengths of sections, and load are varied by different amounts are shown in Figs. 5 and 6. Note that the bounds indicated in Figs. 5 and 6 define the intervals of the stresses as $\sigma_i = [\underline{\sigma}_i, \bar{\sigma}_i]$, where $\underline{\sigma}_i$ is the lower bound and $\bar{\sigma}_i$ is the upper bound on the stress in section i ($i = 1, 2$). The intervals of the parameters are selected as follows:

$$A_i = A_i^0\,[(1 - \delta), (1 + \delta)]; \qquad i = 1, 2 \tag{43}$$

$$E_i = E_i^0\,[(1 - \delta), (1 + \delta)]; \qquad i = 1, 2 \tag{44}$$

$$l_i = l_i^0\,[(1 - \delta), (1 + \delta)]; \qquad i = 1, 2 \tag{45}$$

$$p_1 = [0, 0] \tag{46}$$

$$p_2 = p_2^0\,[(1 - \delta), (1 + \delta)] \tag{47}$$

where the value of δ is varied between 0.0 and 0.01 in Figs. 5 and 6.

2. *Combinatorial Solution*

For the assumed base data (indicated in Sec. V.B.1) and with a value of $\delta = 0.01$ in Eqs. (43–47), the combinatorial approach is used to find the bounds on σ_1 and σ_2 as (49.0099, 51.0101) and (98.0198, 102.0202) psi, respectively.

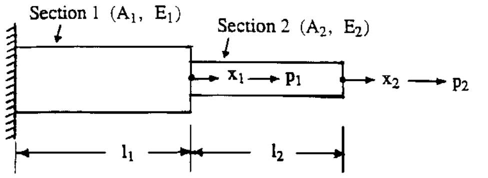

Fig. 4 Stepped bar.

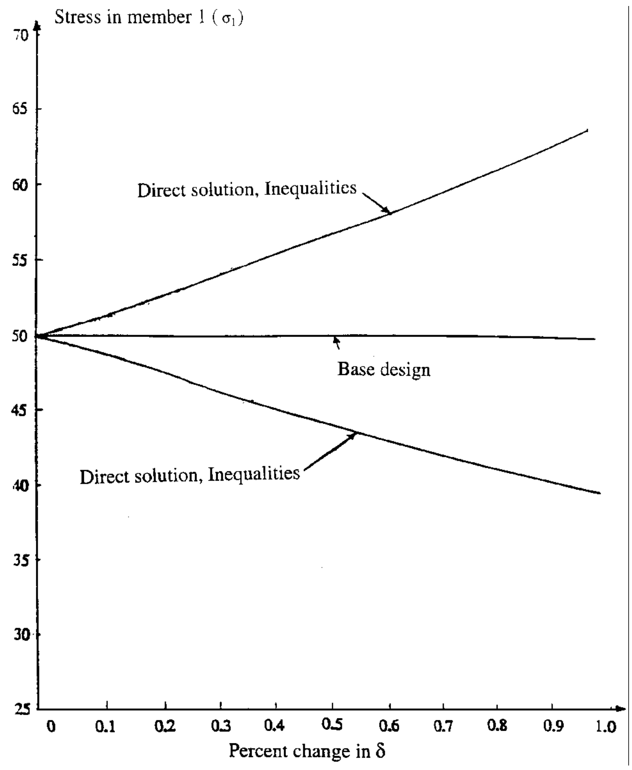

Fig. 5 Bounds on σ_1.

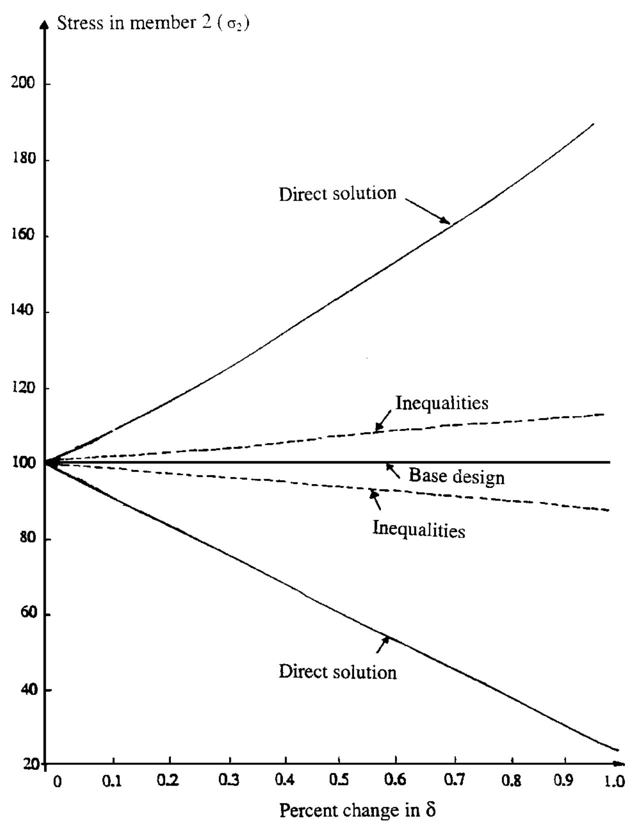

Fig. 6 Bounds on σ_2.

3. Inequality-Based Solution

The inequalities (20) and (21) can be expressed (with $[A] = [K]$ and $\boldsymbol{B} = \boldsymbol{P}$) as

$$[A]\boldsymbol{X} - [D]|\boldsymbol{X}| \leq \bar{\boldsymbol{B}} \tag{48}$$

$$[A]\boldsymbol{X} + [D]|\boldsymbol{X}| \geq \underline{\boldsymbol{B}} \tag{49}$$

where

$$[A] = [a_{ij}], \qquad [D] = [d_{ij}] = [\Delta A], \qquad \underline{\boldsymbol{B}} = \begin{Bmatrix} \underline{b}_1 \\ \underline{b}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \underline{p}_2 \end{Bmatrix}$$

and

$$\bar{\boldsymbol{B}} = \begin{Bmatrix} \bar{b}_1 \\ \bar{b}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \bar{p}_2 \end{Bmatrix}$$

The inequalities (48) and (49) can be rewritten as

$$a_{11}x_1 + a_{12}x_2 + d_{11}|x_1| + d_{12}|x_2| \geq 0 \tag{50}$$

$$a_{21}x_1 + a_{22}x_2 + d_{21}|x_1| + d_{22}|x_2| \geq \underline{p}_2 \tag{51}$$

$$a_{11}x_1 + a_{12}x_2 - d_{11}|x_1| - d_{12}|x_2| \leq 0 \tag{52}$$

$$a_{21}x_1 + a_{22}x_2 - d_{21}|x_1| - d_{22}|x_2| \leq \bar{p}_2 \tag{53}$$

By assuming both x_1 and x_2 to be positive, the vertices of the solution domain, given by the intersection points of the half-spaces defined Eqs. (50–53), can be represented as

$$V_1: \left(x_1 = c_4 x_2, \quad x_2 = \frac{\bar{p}_2}{c_4 c_5 + c_6} \right) \tag{54}$$

$$V_2: \left(x_1 = c_4 x_2, \quad x_2 = \frac{\underline{p}_2}{c_2 c_4 + c_3} \right) \tag{55}$$

$$V_3: \left(x_1 = c_1 x_2, \quad x_2 = \frac{\bar{p}_2}{c_1 c_5 + c_6} \right) \tag{56}$$

$$V_4: \left(x_1 = c_1 x_2, \quad x_2 = \frac{\underline{p}_2}{c_1 c_2 + c_3} \right) \tag{57}$$

where

$$c_1 = -\left(\frac{a_{12} + d_{12}}{a_{11} + d_{11}} \right), \qquad c_2 = a_{21} + d_{21}, \qquad c_3 = a_{22} + d_{22} \tag{58}$$

$$c_4 = \left(\frac{a_{12} - d_{12}}{a_{11} - d_{11}} \right), \qquad c_5 = a_{21} - d_{21}, \qquad c_6 = a_{22} - d_{22} \tag{59}$$

For the case when $A_1^0 = 2$ in.^{2}, $A_2^0 = 1$ in.^{2}, $E_1^0 = E_2^0 = 30 \times 10^6$ psi, $l_1^0 = 10$ in., $l_2^0 = 5$ in., $p_1^0 = 0$, $p_2^0 = 100$ lb, and $\delta = 0.01$, the matrices $[A]$ and $[D]$ are given by

$$[A] = \begin{bmatrix} 0.120048 \times 10^8 & -0.600240 \times 10^7 \\ -0.600240 \times 10^7 & 0.600240 \times 10^7 \end{bmatrix} \tag{60}$$

$$[D] = \begin{bmatrix} 0.360048 \times 10^6 & 0.180024 \times 10^6 \\ 0.180024 \times 10^6 & 0.180024 \times 10^6 \end{bmatrix} \tag{61}$$

so that the inequalities (50–53) reduce to the form

$$11.6448x_1 - 6.1824x_2 \leq 0 \tag{62}$$

$$-6.1824x_1 + 5.8224x_2 \leq 101 \times 10^{-6} \tag{63}$$

$$12.3648x_1 - 5.8224x_2 \geq 0 \tag{64}$$

$$-5.8224x_1 + 6.1824x_2 \geq 99 \times 10^{-6} \tag{65}$$

The four vertices of the solution domain, corresponding to Eqs. (62–65), are shown in Fig. 7. Once the displacement solution is known, the stresses in the two sections of the bar can be computed using interval arithmetic and the results are indicated in Fig. 7. The minimum and maximum values of the stresses σ_1 and σ_2 given by the points V_1, V_2, V_3, and V_4, for different values of δ, are shown in Figs. 5 and 6.

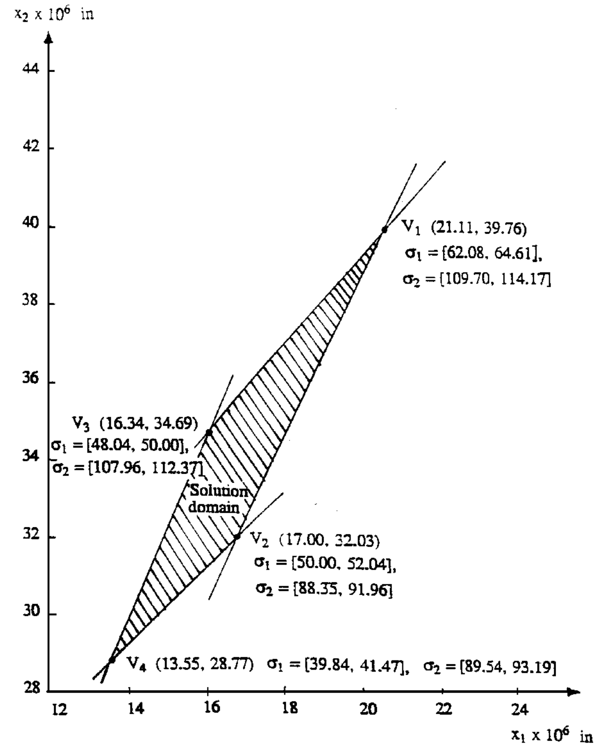

Fig. 7 Solution domain of Eqs. (62–65).

Note that both x_1 and x_2 were assumed positive in solving Eqs. (50–53). In most practical problems, the proper signs of the variables can be determined from the solution of the corresponding crisp or base problem (a crisp problem is one in which the middle values of the interval numbers are used so that the problem can be stated as $[A^0]\boldsymbol{X} = \boldsymbol{B}^0$ with its solution as $\boldsymbol{X}^0$). If one attempts to find the solution of Eqs. (50–53) using improper signs to the variables, no solution can be found. If the solution of the corresponding crisp or base problem is not known to assign suitable signs to the variables, the linear programming approach indicated in Eq. (28) can be used to identify the solution domain of the problem.

4. Growth of Intervals

To illustrate the (reason for the) growth of width of a computed function, the displacement analysis of the stepped bar is considered by treating the area of section 1 (A_1) as the only uncertain parameter. For this case, the displacements of nodes 1 and 2 are given by Eq. (38). Using $A_1 = [\frac{5}{3}, 2]$ in.^{2}, and the base design data for all of the other parameters as indicated earlier, we obtain $k_{11} = [10 \times 10^6, 12 \times 10^6]$, $k_{12} = k_{21} = -6 \times 10^6$, $k_{22} = 6 \times 10^6$, and $p_2 = 100$. In this case, Eq. (38) gives the solution domain as

$$x_1 = \left\{ \left. \frac{100}{k_{11} - 6 \times 10^6} \right|_{k_{11} \in [10 \times 10^6, 12 \times 10^6]} \right\} \tag{66}$$

$$x_2 = \left\{ \left. \frac{k_{11}}{6 \times 10^4 k_{11} - 36 \times 10^{10}} \right|_{k_{11} \in [10 \times 10^6, 12 \times 10^6]} \right\} \tag{67}$$

Because the function

$$\frac{k_{11}}{6 \times 10^4 k_{11} - 36 \times 10^{10}}$$

monotonically decreases in the interval $10 \times 10^6 < k_{11} < 12 \times 10^6$, we obtain

$$x_2 = \left[\frac{1}{30{,}000}, \frac{1}{24{,}000} \right], \qquad x_1 = \left[\frac{1}{60{,}000}, \frac{1}{40{,}000} \right] \tag{68}$$

If we evaluate the expression for x_1 using interval arithmetic, we obtain

$$x_1 = \frac{100}{[10\times 10^6, 12\times 10^6] - 6\times 10^6} = \frac{100}{[4\times 10^6, 6\times 10^6]} = \left[\frac{1}{60{,}000}, \frac{1}{40{,}000}\right] \qquad (69)$$

which can be seen to be exactly the range given in Eq. (68). However, if we evaluate the expression for x_2 using interval arithmetic, we obtain

$$x_2 = \frac{[10\times 10^6, 12\times 10^6]}{(6\times 10^4)\cdot[10\times 10^6, 12\times 10^6] - (36\times 10^{10})} = \left[\frac{1}{36{,}000}, \frac{1}{20{,}000}\right] \qquad (70)$$

which contains the exact range of values of

$$\frac{k_{11}}{6\times 10^4 k_{11} - 36\times 10^{10}}$$

namely,

$$\left[\frac{1}{30{,}000}, \frac{1}{24{,}000}\right]$$

as it should. This shows that the interval arithmetic results in a wider interval than the exact range. The reason is that k_{11} appears twice in the expression of x_2, thereby losing its unique contribution. To verify the validity of this statement, we rewrite the solution of x_2 as

$$x_2 = \frac{1}{6\times 10^4 - [36\times 10^{10}/k_{11}]} \qquad (71)$$

in which the uncertain parameter k_{11} appears only once. The interval arithmetic operations on Eq. (71) lead to

$$x_2 = \frac{1}{6\times 10^4 - (36\times 10^{10})\cdot[1/(10\times 10^6, 12\times 10^6)]} = \left[\frac{1}{30{,}000}, \frac{1}{24{,}000}\right] \qquad (72)$$

which can be seen to be the exact interval given by Eq. (68).

5. Truncation Solution

The interval analysis, coupled with the truncation procedure, is used to determine the bounds on the stresses in the two sections of the bar by varying all of the variables (A_i, E_i, l_i, and p_i, $i = 1, 2$) by $\pm 1\%$ about the base point. The results obtained with different values of the truncation parameter t are shown in Table 1. The maximum and minimum values of σ_1 and σ_2 predicted by the normal interval analysis (with no truncation) and combinatorial approach also are given in Table 1 for comparison. As observed earlier, a value of $t = 0.01$ (maximum relative width of the interval variables) can be seen to predict stresses that are close to those given by the combinatorial approach (exact intervals). With increasing values of t, the stresses converge to those predicted by the normal interval analysis (with no truncation). It also can be observed that the value of t used in the truncation approach is not very critical as long as its value is equal to or greater than the maximum relative deviation present in the input parameters. For example, in Table 1, any value of t in the range 0.01–0.05 predicts the bounds on σ_1 and σ_2 that are reasonably close to those given by the combinatorial approach.

C. Example 3: Ten-Bar Truss

The interval analysis, based on truncation, is considered in finding the stresses in the various members of the 10-bar truss shown in Fig. 8. The base design of the truss is taken to be the same as the minimum weight design of the truss reported by Berke and Khot,[12] and the corresponding data are given in Table 2. The bounds on the member stresses are determined by varying all of the areas of cross section by $\pm 1\%$, and the results are shown in Table 3. In this case, the maximum width of the interval parameters relative to their respective middle values is 0.01. As observed in the previous cases, the truncation parameter $t = 0.01$ is expected to yield accurate results comparable to those predicted by the combinatorial approach. Table 3 shows the results given by the normal interval analysis (with no truncation), combinatorial approach, and the interval analysis with a truncation of $t = 0.01$. As expected, the bounds on the stresses predicted with truncation ($t = 0.01$) are very close to those given by the combinatorial approach. Results were obtained by varying the truncation parameter from $t = 0.05$ to 5.0 (Table 4). These results indicate that the selection of the value of t is not very

Table 1 Effect of truncation: stress analysis of the stepped bar[a]

Method	Bounds on σ_1	Bounds on σ_2
Interval analysis with truncation		
$t = 0.01$	(49.9954, 50.0261)	(99.9908, 100.0530)
$t = 0.02$	(49.9757, 50.0441)	(99.9247, 100.1847)
$t = 0.05$	(48.1693, 51.6406)	(99.5588, 101.4009)
$t = 0.1$	(47.1693, 52.2209)	(97.9647, 102.4009)
$t = 0.2$	(46.2245, 53.3580)	(94.8954, 106.6573)
$t = 0.3$	(39.8405, 64.6134)	(83.3032, 132.9776)
$t = 0.4$	(39.8405, 64.6134)	(75.1058, 141.3383)
$t = 0.5$	(39.8405, 64.6134)	(66.9085, 149.6991)
$t = 0.6$	(39.8405, 64.6134)	(58.7111, 158.0599)
$t = 0.7$	(39.8405, 64.6134)	(50.5138, 166.4207)
$t = 0.8$	(39.8405, 64.6134)	(25.9217, 191.5030)
$t = 0.9$	(39.8405, 64.6134)	(25.9217, 191.5030)
$t = 1.0$	(39.8405, 64.6134)	(25.9217, 191.5030)
Interval analysis with no truncation	(39.8405, 64.6134)	(25.9217, 191.5030)
Combinatorial approach	(49.0099, 51.0101)	(98.0198, 102.0202)

[a]All parameters varied by $\pm 1\%$ about base design.

Table 2 Base design of 10-bar truss

Parameter	Value
Cross-sectional areas of members, in.^{2}	$A_1 = 7.9379$
	$A_i = 0.1$, $i = 2, 5, 6, 10$
	$A_3 = 8.0621$
	$A_4 = 3.9379$
	$A_7 = 5.7447$
	$A_i = 5.569$, $i = 8, 9$
Young's modulus of members, lb/in.^{2}	$E_i = 10^7$, $i = 1, 2, \ldots, 10$
Loads, lb	$p_i = 0$, $i = 1, 2, 3, 5, 6, 7$
	$p_i = -100$, $i = 4, 8$
Stresses in members, lb/in.^{2}	$\sigma_1 = 24.9999$
	$\sigma_2 = 15.5328$
	$\sigma_3 = -25.0001$
	$\sigma_4 = -24.9998$
	$\sigma_5 = 0.0004$
	$\sigma_6 = 15.5328$
	$\sigma_7 = 25.0001$
	$\sigma_8 = -24.9999$
	$\sigma_9 = 24.9999$
	$\sigma_{10} = -21.9668$

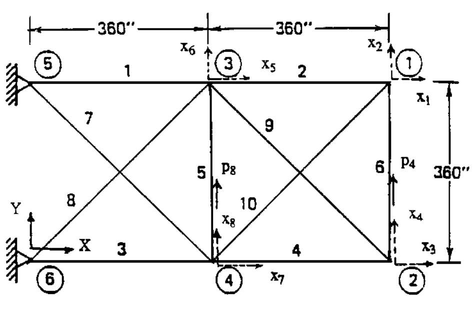

Fig. 8 Ten-bar truss.

Table 3 Effect of truncation: stress analysis of 10-bar truss[a]

Member	Bounds on member stress: Interval analysis with no truncation	Combinatorial analysis	Interval analysis with truncation ($t = 0.01$)
1	(18.7923, 76.3371)	(24.7524, 25.2524)	(24.5471, 25.4692)
2	(−12.9334, 47.8606)	(15.3791, 15.6898)	(15.5247, 15.5440)
3	(−30.9082, −20.1722)	(−25.2526, −24.7526)	(−25.0160, −24.9946)
4	(−62.4723, 9.9812)	(−25.2523, −24.7523)	(−25.0247, −24.9961)
5	(−33.7848, 26.5158)	(0.0004, 0.0004)	(−0.0252, 0.0510)
6	(−76.3371, 107.8070)	(15.3790, 15.6898)	(15.4885, 15.5481)
7	(18.7178, 31.7034)	(24.7526, 25.2526)	(24.9950, 25.0225)
8	(−41.6681, −11.2979)	(−25.2525, −24.7524)	(−25.0110, −24.9909)
9	(−28.0901, 79.3896)	(24.7524, 25.2525)	(24.9647, 25.0267)
10	(−64.6737, 16.6929)	(−22.1887, −21.7493)	(−22.9830, −21.9532)

[a]All member areas changed by ±1%.

Table 4 Interval analysis with truncation: 10-bar truss[a]

Member	$t = 0.05$	$t = 0.1$	$t = 1.0$	$t = 5.0$
1	(23.9647, 26.1281)	(23.7144, 26.8478)	(19.8708, 32.9088)	(19.7537, 33.0749)
2	(15.4911, 15.6531)	(15.5579, 16.0900)	(0.0000, 34.3568)	(−8.0267, 42.4451)
3	(−25.1152, −24.9825)	(−27.4036, −23.0721)	(−28.0178, −22.6486)	(−28.2994, −22.3981)
4	(−25.1655, −24.9372)	(−25.6893, −24.6984)	(−51.9876, 0.0000)	(−54.2850, 2.2456)
5	(−0.1307, 0.2249)	(−1.2217, 0.4084)	(−6.5735, 0.0000)	(−19.9724, 13.3149)
6	(15.2158, 15.7301)	(15.1910, 15.6587)	(0.0000, 31.5214)	(−62.2475, 93.7663)
7	(23.8040, 26.2700)	(22.6592, 27.4025)	(21.4080, 28.9116)	(21.1390, 29.1893)
8	(−26.1337, −23.8956)	(−25.5695, −24.8556)	(−38.8648, −13.6849)	(−39.1612, −13.4303)
9	(24.8961, 25.1457)	(24.1199, 26.1126)	(0.0000, 51.0626)	(−19.5445, 70.6318)
10	(−22.1908, −21.8103)	(−23.2532, −21.3310)	(−47.4041, 0.0000)	(−56.4844, 9.0173)

[a]Areas changed by ±1%.

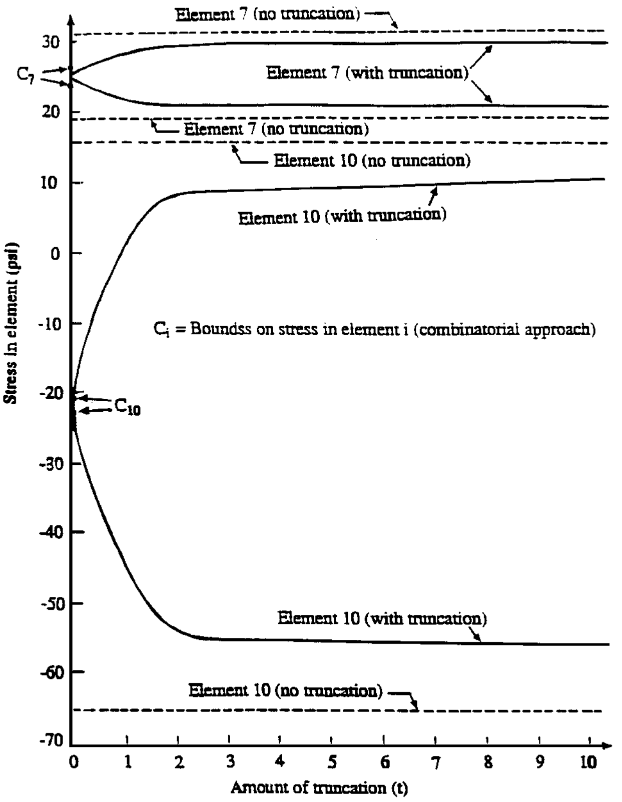

Fig. 9 Variation of bounds on stress with truncation (elements 7 and 10).

critical in predicting reasonably accurate stresses. For example, any value between $t = 0.01$ and 0.05 predicts the member stresses that are close to those predicted by the combinatorial approach (exact intervals). It also can be observed that the maximum and minimum values of the stresses tend to converge to the corresponding values given by the normal interval analysis (with no truncation) as t increases (see Tables 3 and 4). The convergence of the stresses in members 7 and 10 is shown graphically in Fig. 9.

VI. Conclusions

1) The analysis of uncertain structural systems, where the parameters are described in terms of intervals, is considered. Several methods, namely, the direct method, Gaussian elimination approach, a combinatorial approach, an inequality-based method, and a truncation approach, are presented.

2) It is found that the method based on inequalities is more accurate than the direct method in identifying the solution domain.

3) The interval analysis proves to be a viable, economical, and feasible alternative for finding the extreme values of structural response (such as displacements and stresses). Traditionally, the combinatorial approach is used for this purpose. However, this will be a computationally tedious and expensive procedure. For example, even for a simple problem such as the 10-bar truss considered in example 4, if the areas of cross section, Young's moduli, and lengths of individual members are varied independently (defining a range for each parameter), the combinatorial approach requires 2^{30} analyses to find the maximum and minimum values of the stresses. On the other hand, once the computer code is developed, the interval analysis requires an equivalent of only two or three analyses.

4) It is observed that the direct, the Gaussian elimination, and the inequality-based approaches tend to overestimate the extreme values of the response quantities, especially with increasing widths (ranges) and number of the uncertain input parameters. However, the procedure can predict the extreme values of response accurately when the widths and number of uncertain parameters are small. The overestimation of the extreme values of the response quantities can sometimes be useful for a conservative design.

5) When the widths or ranges of the uncertain parameters are large, the truncation procedure can be used to obtain reasonable estimates of the largest and smallest values of the response quantity. It was found that a value of the truncation parameter t equal to the maximum relative width of the interval numbers appearing in the problem data gives reasonably accurate results (which are comparable to the exact intervals given by the combinatorial approach).

6) The interval analysis methods presented in this work are expected to be useful not only in predicting the extreme values of response parameters, but also in optimization, fuzzy-logic-based analysis, and probabilistic analysis of structures.

Appendix A: Interval Arithmetic

The interval arithmetic operations on real numbers can be defined as follows[1,2]:

$$[\underline{a}, \bar{a}] * [\underline{b}, \bar{b}] = \{x * y \mid \underline{a} \le x \le \bar{a}, \underline{b} \le y \le \bar{b}\} \tag{A1}$$

where $*$ denotes an arithmetic operation such as $+$, $-$, $\times$ (or $\cdot$) or $\div$ (or $/$). The division operation $[\underline{a}, \bar{a}]/[\underline{b}, \bar{b}]$ is not defined if $0 \in [\underline{b}, \bar{b}]$. More explicitly, Eq. (A1) can be expressed as

$$[\underline{a}, \bar{a}] + [\underline{b}, \bar{b}] = [\underline{a} + \underline{b}, \bar{a} + \bar{b}] \tag{A2}$$

$$[\underline{a}, \bar{a}] - [\underline{b}, \bar{b}] = [\underline{a} - \bar{b}, \bar{a} - \underline{b}] \tag{A3}$$

$$[\underline{a}, \bar{a}] \cdot [\underline{b}, \bar{b}] = [\min(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b}), \quad \max(\underline{a}\underline{b}, \underline{a}\bar{b}, \bar{a}\underline{b}, \bar{a}\bar{b})] \tag{A4}$$

$$[\underline{a}, \bar{a}]/[\underline{b}, \bar{b}] = [\underline{a}, \bar{a}] \cdot [1/\bar{b}, 1/\underline{b}] \qquad \text{if} \qquad 0 \notin [\underline{b}, \bar{b}] \tag{A5}$$

Note that a real number a is denoted by a degenerate interval $[a, a]$. Equation (A1) indicates that interval addition and interval multiplication are both associative and commutative. Although the distributive law does not hold true always, the subdistributive law and the inclusion monotonicity law hold true. The interval arithmetic operations can be extended to matrix computations.[1,2] For example, if $[A] = [a_{ij}]$ is a $p \times q$ interval matrix and $[B] = [b_{ij}]$ is a $q \times r$ interval matrix, where $a_{ij} = [\underline{a}_{ij}, \bar{a}_{ij}]$ and $b_{ij} = [\underline{b}_{ij}, \bar{b}_{ij}]$ are interval numbers, then the following are true:

$$[A][B] = [C] = [c_{ij}] \tag{A6}$$

where

$$c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}; \qquad i = 1, 2, \ldots, p; \qquad j = 1, 2, \ldots, r$$

$$\{[A'][B'] \mid [A'] \in [A], [B'] \in [B]\} \subseteq \{[C'] \mid [C'] \in [A][B]\} \tag{A7}$$

where the equality, in general, does not hold true.

Appendix B: Sample Calculations

To illustrate the basic computations involved, the following interval equations are considered:

$$\begin{bmatrix} (2, 4) & (4, 6) \\ (0, 2) & (5, 7) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} (24, 28) \\ (23, 27) \end{Bmatrix} \tag{B1}$$

Noting that, for the crisp equations,

$$[A]\boldsymbol{X} = \boldsymbol{B} \tag{B2}$$

or

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \end{Bmatrix} \tag{B3}$$

the inverse of $[A]$ is given by

$$[A]^{-1} = \frac{1}{a_{12}\, a_{21} - a_{11}\, a_{22}} \begin{bmatrix} -a_{22} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix} \tag{B4}$$

the solution vector can be determined as

$$\boldsymbol{X} = [A]^{-1} \boldsymbol{B} \tag{B5}$$

For the interval coefficient matrix of Eq. (B1), the inverse can be found, using interval arithmetic operations of Eq. (B4), as

$$[A]^{-1}_{\text{int arith}} = \begin{bmatrix} (-3.5, 0.25) & (-168.0, 12.0) \\ (-0.07143, 1.0) & (-8.0, 112.0) \end{bmatrix} \tag{B6}$$

When a combinatorial approach is used, Eq. (B4) gives

$$[A]^{-1}_{\text{combi}} = \begin{bmatrix} (-2.5, 3.5) & (-3.0, 3.0) \\ (-1.0, 1.0) & (-1.0, 1.0) \end{bmatrix} \tag{B7}$$

Thus the solution vector can be found as follows.

1) Using interval arithmetic [Eqs. (B6) and (B5)],

$$\boldsymbol{X} = \begin{Bmatrix} (-4633.9995, 331.0) \\ (-218.0, 3051.9997) \end{Bmatrix} \tag{B8}$$

2) Using combinatorial approach [Eqs. (B7) and (B5)],

$$\boldsymbol{X} = \begin{Bmatrix} (-5.25, 29.0) \\ (-5.0, 7.5) \end{Bmatrix} \tag{B9}$$

As expected, the ranges of components of $\boldsymbol{X}$ in Eq. (B8) can be seen to be wider than those in Eq. (B9).

Acknowledgments

The first author is grateful for the support he received through Grant 9301992-MSS from the National Science Foundation (Program Director: Ken P. Chong) and also the summer faculty fellowship he received at NASA Lewis Research Center in the NASA–Ohio Aerospace Institute Collaborative Aerospace Research and Fellowship Program during the summer of 1995.

References

[1]Moore, R. E., *Interval Analysis*, Prentice–Hall, Englewood Cliffs, NJ, 1966, pp. 25–39.

[2]Neumaier, A., *Interval Methods for Systems of Equations*, Cambridge Univ. Press, Cambridge, England, UK, 1990, pp. 77–169.

[3]Rao, S. S., "Description and Optimum Design of Fuzzy Mechanical Systems," *Journal of Mechanisms, Transmissions, and Automation in Design*, Vol. 109, No. 1, 1987, pp. 126–132.

[4]Rao, S. S., "Multiobjective Optimization in Structural Design in the Presence of Uncertain Parameters and Stochastic Process," *AIAA Journal*, Vol. 22, No. 11, 1984, pp. 1670–1678.

[5]Rao, S. S., and Sawyer, J. P., "Fuzzy Finite Element Approach for the Analysis of Imprecisely Defined Systems," *AIAA Journal*, Vol. 33, No. 12, 1995, pp. 2364–2370.

[6]Rao, S. S., *Reliability-Based Design*, McGraw–Hill, New York, 1992, Chap. 10.

[7]Gay, D. M., "Solving Interval Linear Equations," *SIAM Journal on Numerical Analysis*, Vol. 19, No. 4, 1982, pp. 858–870.

[8]Rohn, J., "Systems of Linear Interval Equations," *Linear Algebra and Its Application*, Vol. 126, Dec. 1989, pp. 39–78.

[9]Chen, R., and Ward, A. C., "Introduction to Interval Matrices in Design," *Design Theory and Methodology*, DE-Vol. 42, American Society of Mechanical Engineers, New York, 1992, pp. 221–227.

[10]Oettli, W., "On the Solution Set of a Linear System with Inaccurate Coefficients," *SIAM Journal on Numerical Analysis*, Series 8, Vol. 2, No. 1, 1965, pp. 115–118.

[11]Oettli, W., and Prager, W., "Compatibility of Approximate Solution of Linear Equations with Given Error Bounds for Coefficients and Right-Hand Sides," *Numerische Mathematik*, Vol. 6, 1964, pp. 405–409.

[12]Berke, L., and Khot, N. S., "Use of Optimality Criteria Methods for Large Scale Systems," *Structural Optimization*, AGARD Lecture Series No. 70, 1974, pp. 1.1–1.29.

R. K. Kapania
Associate Editor